Problem 1
Let $S = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$. Prove that S has Jordan measure 0.

Problem 2
Show that an unbounded set cannot have Jordan measure 0.

Problem 3
For each of the following functions, find an anti-derivative. The anti-derivative must be defined wherever the function is defined.

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$.
2. $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin x - \cos x$
3. $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{x^2}$
4. $f : \mathbb{R} \to \mathbb{R}, f(x) = |x|$
5. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^n$, where $n \in \mathbb{Z}$ and $n \neq -1$. (What goes wrong when $n = -1$? We'll explore this in Problem 4)

Problem 4
Let $f : [0, \infty) \to \mathbb{R}$ be the function $f(x) = x^3$. Let $F : [0, \infty) \to \mathbb{R}$ be the function:

$$F(x) = \int_0^x t^3 dt.$$

1. Assume $h \in \mathbb{R}$. Without using FTC and instead using algebra and other things we know about integration, simplify the expression:

   $$\frac{F(x+h) - F(x)}{h}$$

   into a single real number times a single integral.
2. Show that:

   $$\lim_{h \to 0} \left(\frac{F(x+h) - F(x)}{h} \right) = f(x)$$

   and use this to conclude that F is an anti-derivative of f. *Hint: you don't need anything fancy here, use the fact that f is monotone increasing.*
3. If G is any other anti-derivative of f, show that $F(x) = G(x) - G(0)$.
4. Using (3) and the anti-derivative found in Problem 2, find

   $$\int_{2021}^{2022} t^3 dt.$$

   Simplify your answer so that it contains no integral sign.

Problem 5

Let $L : (0, \infty) \to \mathbb{R}$ be defined by

$$L(x) = \int_1^x \frac{1}{t} dt.$$

1. Using a similar argument to Problem 3, i.e not using FTC, show that L is an anti-derivative of $\frac{1}{x}$ on the domain $(0, \infty)$. In this case, L is the unique anti-derivative of $\frac{1}{x}$ satisfying $L(1) = 0$.
2. Show that $L(xy) = L(x) + L(y)$ for $x, y \in \mathbb{R}$. *Hint: treat y as a constant, differentiate with respect to x, or the other way around.*
3. Show that $L(1/x) = -L(x)$ for any $x \in \mathbb{R}$, again using differentiation.
4. Show that $L(x^n) = nL(x)$ for any integer $n \neq -1$ and $x \in \mathbb{R}$.
5. Show that L is strictly increasing. Combining this with the fact that it's differentiable with continuous derivative, we can apply IFT to obtain a C^1 inverse $\exp : \mathbb{R} \to (0, \infty)$. Do you recognize this function and its inverse?